$$E \begin{cases} \nearrow \mathbb{F}_q((\pi)) \\ \searrow [E:\mathbb{Q}_p] < +\infty,\ \mathcal{O}_E/\pi = \mathbb{F}_q \end{cases}$$

$\mathrm{Perf}_{\mathbb{F}_q}$ = $\mathbb{F}_q$ - perfectoid spaces + pro - étale topology

Ψ $\quad$ $\mathrm{Spa}(\mathbb{F}_q)$ = final object of the pro - étale topos
not representable

S

$\rightsquigarrow X_S$ = E - adic space

= " family of curves $(X_{k(s),k(s)^+})_{s \in S}$ "

$\mathcal{P}ic$ = Picard stack on $\mathrm{Perf}_{\mathbb{F}_q}$

$\mathcal{P}ic(S) = \{$ Line bundles on $X_S \}$ (groupoid of)

$$\mathcal{P}ic = \coprod_{d \in \mathbb{Z}} \underbrace{\mathcal{P}ic^d}_{\text{open/closed substack}}$$

Classifying stacks of pro-étale $\underline{E}^\times$-torsors

Th: $\mathrm{Pic}^d \xrightarrow{\sim} [\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^\times]$

$\mathrm{Pic}^d(S) \ni \mathcal{L} \longmapsto [T/S \longmapsto \mathrm{Isom}(\mathcal{O}(d), \mathcal{L}|_{X_T})]$

Div^1:

Untilts: **Def:** $\mathcal{F}$ pro-étale sheaf on ~~...~~ Perf_E. Let $\mathcal{F}^\diamond$ = pro-étale $\boxed{\text{sheaf}}$ on $\mathrm{Perf}_{\mathbb{F}_q}$ be defined by

$$\mathcal{F}^\diamond(S) = \left\{ (S^\#, \iota, x) \;\middle|\; \begin{array}{l} S^\# \in \mathrm{Perf}_{\mathbb{F}_q} \\ \iota : S \xrightarrow{\sim} S^{\#,\flat}, \quad x \in \mathcal{F}(S^\#) \end{array} \right\} / \sim$$

Consequence of $\forall T \in \mathrm{Perf}_E$ iso. of pro-étale sites

$$(-)^\flat : \mathrm{Perf}_T \xrightarrow{\sim} \mathrm{Perf}_{T^\flat}$$

②

Ex: $\mathrm{Spa}(E)^\diamond$ = sheaf of untilts over E.

$\mathrm{Spa}(E)^\diamond(S) = \{S^\#, \iota\}/\sim$ pro-étale topos

$\left[(-)^\diamond : \widetilde{\mathrm{Perf}_E} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_q}} / \mathrm{Spa}(E)^\diamond \right]$ localized topos

Def: R perfectoid $\mathbb{F}_q$-algebra.

$\xi \in W_{\mathcal{O}_E}(R^\circ)$, resp. $R^\circ[[\pi]]$, is primitive of degree 1

$\xi = \sum_{n \geq 0} [x_n]\pi^n$, if $x_0 \in R^{\circ\circ} \cap R^\times$ and $x_1 \in (R^\circ)^\times$.

Prop: R perfectoid E-algebra

$$\theta : W_{\mathcal{O}_E}(R^{b,\circ}) \longrightarrow R^\circ$$

$$\sum_{n \geq 0} [x_n]\pi^n \longmapsto \sum_{n \geq 0} x_n^\# \pi^n$$

$x \in R^b$, $x = (x^{(n)})_{n \geq 0}$

$x^{(n)} \in R$, $(x^{(n+1)})^p = x^{(n)}$

$x^\# := x^{(0)}$

is surjective with ker θ generated by a deg. 1 primitive element.

(2) Reciprocally, if R = perfectoid $\mathbb{F}_q$-algebra then

$$\underbrace{W_{\mathcal{O}_E}(R^\circ)\left[\tfrac{1}{u}\right]/(\xi)}\ = \text{perfectoid } E\text{-algebra that is}$$

deg. 1 primitive element

an untilt of R via

$$R \ni x \longmapsto \left([x^{1/q^n}] \bmod \xi\right)_{n\geq 0} \in (\ \cdot\ )^\flat$$

Thus. R = perfectoid $\mathbb{F}_q$-algebra

$$\left[\{\text{untilts of } R \text{ over } E\} \simeq \left\{\begin{matrix}\text{deg. 1 primitive}\\ \text{elements}\end{matrix}\right\} / W_{\mathcal{O}_E}(R^\circ)^\times\right]$$

$\Rightarrow S \in \mathrm{Perf}_{\mathbb{F}_q}$

$$\{\text{untilts of } S\} \simeq \left\{\begin{matrix}\boxed{\text{Cartier divisors}}\ T \hookrightarrow Y_{S'} \\ \text{locally on } S \text{ defined by } V(\xi)\end{matrix}\right\}$$

deg. 1 primitive

$\Downarrow$

ξ regular on $\mathcal{O}_{Y_{S'}}$

③

Def: $d \geq 1$. $\underbrace{\mathrm{Div}^d}_{}$ = pro-étale sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ defined

sheaf of deg. d effective Cartier divisors

by $\mathrm{Div}^d(S) = \{(\mathcal{L}, u) \mid \mathcal{L}/X_S$ line bundle, $u \in H^0(X_S, \mathcal{L})$
s.t. $\forall s \in S$, $u_{|X_{k(s), k(s)^+}} \neq 0 \} / \sim$

One checks that if $S^\# =$ untilt of S then

$$\begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr] & Y_S \arrow[d] \\ & X_S \end{tikzcd}$$

injective $\Rightarrow$ defines a deg. 1 Cartier divisor on X_S.

This defines a morphism

$$\begin{tikzcd} \mathrm{Spa}(E)^\diamond \arrow[rr] \arrow[dr] & & \mathrm{Div}^1 \\ & \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} \arrow[ur] & \end{tikzcd}$$

factorization

Prop: $\mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1$

Rem: * $\begin{array}{c} X_S \\ \downarrow \\ \mathrm{Spa}(E) \end{array}$ satisfies $X_S^\diamond = (S \times \mathrm{Spa}(E)^\diamond)/\varphi_S^{\mathbb{Z}}$

Thus:

$$\begin{array}{c} X_S^\diamond = (S \times \mathrm{Spa}(E)^\diamond)/\varphi_S^{\mathbb{Z}} \\ \downarrow \\ \mathrm{Spa}(E)^\diamond \end{array} \quad \Bigg| \quad \begin{array}{c} \mathrm{Div}^1_S = (S \times \mathrm{Spa}(E)^\diamond)/\varphi_{E^\diamond}^{\mathbb{Z}} \\ \downarrow \\ S \end{array}$$

$\varphi_S \circ \varphi_{E^\diamond} =$ absolute Frob. of $S \times \mathrm{Spa}(E)^\diamond$ acts trivially on étale site or topological space

$$\Rightarrow \quad |X_S| = |X_S^\diamond| = |\mathrm{Div}^1_S| \to |S|$$

, X_S and Div^1_S have the same étale site and top. space but are not isomorphic!

* Div^1 is a diamond but it is not quasi-separated (not spatial in particular).

The good thing to consider is Div^1 ← representable in spatial diamonds

$$\begin{array}{c} \mathrm{Div}^1 \\ \downarrow \\ \underbrace{\mathrm{Spa}(\mathbb{F}_q)} \end{array}$$

final object of the pro-étale topos, not representable, not quasi-compact

<u>Th.</u> $\forall d \geq 1$, Div^d is a diamond and $\Sigma^d : (\mathrm{Div}^1)^d \to \mathrm{Div}^d$ is quasi-pro-étale surjective. This induces an isomorphism of diamonds

$$(\mathrm{Div}^1)^d / \mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d.$$

← pro-étale quotient

<u>quasi-pro-étale</u>: $f : X \to Y$ is quasi-pro-étale if pro-étale locally on X and Y it is pro-étale.

<u>Prop</u> (Scholze): X and Y qc qs. Then f is quasi-pro-étale iff its geometric fibers are pro-finite.

Ex: $\mathrm{Spa}\left(\mathbb{C}_p\langle T^{1/p^\infty}\rangle\right) \longrightarrow \mathrm{Spa}\left(\mathbb{C}_p\langle T^{1/p^\infty}\rangle\right)$

$$t \longmapsto t^n$$

Kummer map is quasi-pro-étale with finite fibers but not étale.

$\Rightarrow$ it is a pro-étale épimorphism

Mainly a consequence of our main theorem with Fontaine:

if $P = \bigoplus_{d \geq 0} \underbrace{B_F^{\varphi = \pi^d}}_{P_d}$ associated to $F|\mathbb{F}_q$ perfectoid alg. closed then $\forall d \geq 2$ $\forall x \in P_d$ $\exists y_1, \dots, y_d \in P_1$ such that $x = y_1 \cdots y_d$.

Local systems on Div^1:

[Now everything is over $\overline{\mathbb{F}}_q$ i.e. we work in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$.]

[**Prop:** $\overline{\mathbb{Q}}_\ell$-local systems over $\mathrm{Div}^1 \simeq \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$.]

i.e. "$\pi_1(\mathrm{Div}^1) = W_E$".

$$\to \mathrm{Div}^1_{\overline{\mathbb{F}}_q} = \underbrace{\left(\mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\overline{\mathbb{F}}_q) \right)}_{\mathrm{Spa}(\breve{E})^\diamond} \Big/ \varphi_{E^\diamond}^{\mathbb{Z}}$$

$\mathrm{Spa}(\breve{E})^\diamond$ where $\breve{E} = \widehat{E^{un}}$

$\sigma \curvearrowright$

σ = Frobenius

$$\varphi_{E^\diamond} \circ \varphi_{\overline{\mathbb{F}}_q} = \text{absolute Frob.}$$

$$\Downarrow$$

$\overline{\mathbb{Q}}_\ell$ - loc. systems on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$ = $\underset{\sigma}{\overset{\|}{\varphi_{\overline{\mathbb{F}}_q}}}$ - eq. $\overline{\mathbb{Q}}_\ell$ - loc. systems on $\mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(\overline{\mathbb{F}}_q) = \mathrm{Spa}(\breve{E})^\diamond$

$= \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$ $\square$

Thus, if $\varphi : W_E \to \overline{\mathbb{Q}}_\ell^\times$ character $\rightsquigarrow$ $\mathcal{E}$ = rank 1 $\overline{\mathbb{Q}}_\ell$-local system on Div^1

Via the preceding theorem: $\mathrm{Div}^d = (\mathrm{Div}^1)^d / \sigma_d^\diamond$ ← pro-étale quotient

one verifies that $\mathcal{E}^{(d)} := \left(\Sigma_*^d \, \mathcal{E}^{\boxtimes d} \right)^{\sigma_d}$ is a rank 1 $\overline{\mathbb{Q}}_\ell$ - local system on Div^d!

→ Completes the first part of the construction.